Derivation of an alternative expression for process noise in MKS

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Sweung Cheung and K. Clint Slatton

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Point of Contact:
Prof. K. Clint Slatton
University of Florida; PO Box 116130; Gainesville, FL 32611
Tel: 352.392.0634, Fax: 352.392.0044, E-mail: slatton@ece.ufl.edu
**Problem statement:**

In early spring 2004, Sweung Cheung derived a new expression for computing process noise $Q$ in the standard Multiscale Kalman Smoother MKS algorithm.
Derivation of an alternative expression for $Q(s)$

04-Feb-04
By Sweungwon Cheung

Forward (Down-sweep) Model

Let the spatial interval be $0 \leq s \leq M$ and spatial state model is

$$
x(s) = \Phi(s)x(Bs) + \Gamma(s)w(s)$$
$$y(s) = H(s)x(s) + v(s)$$

(1)

where $w(s)$ is white process and uncorrelated $x(0)$

$$E[w(s)] = 0 \quad E[w(s)w^T(t)] = I\delta(s-t)$$
$$E[x(0)] = 0 \quad E[x(0)x^T(0)] = P_x(0)$$

and forward orthogonality is satisfied, i.e. $E[x(0)w^T(s)] = 0$ for $s \geq 0$

Backward (Up-sweep) Model

The backward model can be get from (1) by just reversing the direction of spatial.

$$x(Bs) = \Phi^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)w(s)$$
$$y(s) = H(s)x(s) + v(s)$$

(2)

The backward state process is still Markove. But it does not satisfy the backward orthogonality between $x(M)$ and $w(s)$ for $s \leq M$.

We can define $w(s)$ as following by Markovity

$$w(s) = E[w(s) | x(s), x(s+1),...,x(M)] + \bar{w}(s)$$
$$= E[w(s) | x(s)] + \bar{w}(s)$$
where $\mathbb{E}[w(s)|x(s)]$ is MMSE estimate

Since $\tilde{w}(s) = w(s) - \mathbb{E}[w(s) | x(s)]$ so that

$$\tilde{w}(s) \perp x(s)$$

We assumed that $w(s)$ and $x(s)$ are zero mean Gaussian so that we can find the following equation (3) p324

$$\mathbb{E}[w(s) | x(s)] = \mathbb{E}[w(s)x^T(s)]\mathbb{E}[x(s)x^T(s)]^{-1}x(s) \quad (3)$$

Since $x(s) = \Phi(s)x(Bs) + \Gamma(s)w(s)$ and we can substitute $x(s)$ into (3) so that

$$\mathbb{E}[w(s) | x(s)] = \Gamma^T(s)\mathbb{P}_{x}^{-1}(s)x(s)$$

The backward markov model can be rewritten [1]

$$x(Bs) = \Phi^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)\left[\mathbb{E}[w(s) | x(s)] + \tilde{w}(s)\right]$$

$$= \Phi^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\mathbb{P}_{x}^{-1}(s)x(s) - \Phi^{-1}(s)\Gamma(s)\tilde{w}(s)$$

$$= \Phi^{-1}(s)\left[I - \Gamma(s)\Gamma^T(s)\mathbb{P}_{x}^{-1}(s)\right]x(s) - \Phi^{-1}(s)\Gamma(s)\tilde{w}(s)$$

Let

$$F(s) = \Phi^{-1}(s)\left[I - \Gamma(s)\Gamma^T(s)\mathbb{P}_{x}^{-1}(s)\right]$$

$$\overline{w}(s) = -\Phi^{-1}(s)\Gamma(s)\tilde{w}(s)$$

As a result, the backward markov model is

$$x(Bs) = F(s)x(s) + \overline{w}(s)$$

$$y(s) = H(s)x(s) + v(s)$$

Since

$$\mathbb{P}_{x}(s) = \Phi(s)\mathbb{P}_{x}(Bs)\Phi^T(s) + \Gamma(s)\Pi^T(s)$$

$$\Phi^{-1}(s)\mathbb{P}_{x}(s) = \Phi^{-1}(s)\Phi(s)\mathbb{P}_{x}(Bs)\Phi^T(s) + \Phi^{-1}(s)\Gamma(s)\Pi^T(s)$$

$$\Phi^{-1}(s) = \Phi^{-1}(s)\Phi(s)\mathbb{P}_{x}(Bs)\Phi^T(s)\mathbb{P}_{x}^{-1}(s) + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\mathbb{P}_{x}^{-1}(s)$$

$$\Phi^{-1}(s) - \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\mathbb{P}_{x}^{-1}(s) = \mathbb{P}_{x}(Bs)\Phi^T(s)\mathbb{P}_{x}^{-1}(s)$$

so that

$$F(s) = \mathbb{P}_{x}(Bs)\Phi^T(s)\mathbb{P}_{x}^{-1}(s)$$

Let $Q(s) = \mathbb{E}[\overline{w}(s)\overline{w}^T(s)]$ and
\[
E[\tilde{w}(s)\tilde{w}^T(s)] = E[(w(s) - E[w(s) | x(s))](w(s) - E[w(s) | x(s)])^T] \\
= E[(w(s) - \Gamma^T(s)P_x^{-1}(s)x(s))(w(s) - \Gamma^T(s)P_x^{-1}(s)x(s))^T] \\
= 1 - \Gamma^T(s)P_x^{-1}(s)E[x(s)x(s)^T]P_x^{-1}(s)\Gamma(s) \\
= 1 - \Gamma^T(s)P_x^{-1}(s)P_x(s)P_x^{-1}(s)\Gamma(s) \\
= 1 - \Gamma^T(s)P_x^{-1}(s)\Gamma(s)
\]

So that

\[
Q(s) = E[\tilde{w}(s)\tilde{w}^T(s)] \\
= \Phi^{-1}(s)\Gamma(s)E[\tilde{w}(s)^T(s)]\Gamma^T(s)\Phi^{-T}(s) \\
= \Phi^{-1}(s)\Gamma(s)[1 - \Gamma^T(s)P_x^{-1}(s)\Gamma(s)]\Gamma^T(s)\Phi^{-T}(s) \\
= \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) - \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s)
\]

Since

\[
P_x(s) = \Phi(s)P_x(Bs)\Phi^T(s) + \Gamma(s)\Gamma^T(s)
\]

\[
\Phi^{-1}(s)P_x(t) = \Phi^{-1}(t)\Phi(s)P_x(Bs)\Phi^T(s) + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)
\]

\[
\Phi^{-1}(s) = \Phi^{-1}(s)\Phi(s)P_x(Bs)\Phi^T(s)P_x^{-1}(s) + \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x^{-1}(s)
\]

\[
\Phi^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) = \Phi^{-1}(s)\Phi(s)P_x(Bs)\Phi^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \\
+ \Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s)
\]

\[
\Phi^{-1}(s)\Gamma(s)\Gamma^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) = \Phi^{-1}(s)\Phi(s)P_x(Bs)\Phi^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) + P_x(Bs)\Phi^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s)
\]

\[
Q(s) \text{ can be rewritten as} \\
Q(s) = P_x(Bs)\Phi^T(s)P_x^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \quad (*)
\]

Since

\[
F(s) = P_x(Bs)\Phi^T(s)P_x^{-1}(s)
\]

and

\[
P_x(s) = \Phi(s)P_x(Bs)\Phi^T(s) + \Gamma(s)\Gamma^T(s)
\]

\[
P_x(s)\Phi^T(s) = \Phi(s)P_x(Bs)\Phi^T(s)\Phi^{-T}(s) + \Gamma(s)\Gamma^T(s)\Phi^{-T}(s) \\
\Gamma(s)\Gamma^T(s)\Phi^{-T}(s) = P_x(t)\Phi^{-T}(s) - \Phi(s)P_x(Bs)
\]

so that

\[
Q(s) = F(s)\left(P_x(s)\Phi^{-T}(s) - \Phi(s)P_x(Bs)\right) \\
= F(s)P_x(s)\Phi^{-T}(s) - F(s)\Phi(s)P_x(Bs)
\]

Since

\[
F(s) = P_x(Bs)\Phi^T(s)P_x^{-1}(s) \rightarrow F(s)P_x(s)\Phi^{-T}(s) = P_x(Bs)
\]

so that
\[ Q(s) = P_\chi(Bs) - F(s)\Phi(s)P_\chi(Bs) \]
\[ = (I - F(s)\Phi(s))P_\chi(Bs) \]

The equation of \( Q(s) \) which was written in Dr. Slatton’s handout can be derived from (*)

\[
Q(s) = P_\chi(Bs)\Phi^T(s)P_\chi^{-1}(s)\Gamma(s)\Gamma^T(s)\Phi^{-T}(s)
\]
\[
= [P_\chi(Bs)\Phi^T(s)P_\chi^{-1}(s)P_\chi(s)\Phi^{-T}(s) - \Phi(s)P_\chi(Bs)]
\]
\[
= [P_\chi(Bs) - P_\chi(Bs)\Phi^T(s)P_\chi^{-1}(s)\Phi(s)P_\chi(Bs)]
\]
\[
= P_\chi(Bs)[I - \Phi^T(s)P_\chi^{-1}(s)\Phi(s)P_\chi(Bs)]
\]

As a result, we can get two equations for \( Q(s) \)

\[
Q(s) = P_\chi(Bs)[I - \Phi^T(s)P_\chi^{-1}(s)\Phi(s)P_\chi(Bs)]
\]
\[
= (I - F(s)\Phi(s))P_\chi(Bs)
\]

In my opinion, the second equation has advantages like no inverse matrix and less computation complexity.
References

